Fibers of Spec maps

For $P \subseteq R$ a prime ideal, define the <u>residue field</u> of P to be

$$k(P) := R_P / PR_P = (R/P)_P = K(R/P)$$

Remark: The second equality holds because localization commutes w/ taking quotients, since it is flat:

If $N \subseteq M$, then localizing

$$0 \to N \to M \to M/N \to 0$$

we get the SES

$$0 \to N_P \to M_P \to (M/N)_P \to 0$$

$$\Rightarrow M_P / N_P \cong (M/N)_P.$$

There is a canonical map $R \to k(P)$ given by

$$R \to R_P \to R_P/(PR_P) = k(P)$$

What is the induced map on Spec?

$\operatorname{Spec} k(P) = \{(0)\}$, and the kernel of $R \to k(P)$ is P.
So $\{(0)\} \mapsto P$.

Theorem: Let $\varphi: R \to S$ be a ring map and

$f: \operatorname{Spec} S \to \operatorname{Spec} R$ the induced map on spec.
For each $P \in \operatorname{Spec} R$, there is a natural bijection between the fiber $f^{-1}(P) \subseteq \operatorname{Spec} S$ and $\operatorname{Spec}\left(k(P) \otimes_R S\right)$.

Pf: First note that $f^{-1}(P) = \{ Q \in \operatorname{Spec} S \mid \varphi^{-1}(Q) = P \}$.

Let $U = \varphi(R \setminus P) \subseteq S$. U is a multiplicative set since $R \setminus P$ is: $\varphi(a)\varphi(b) = \varphi(ab)$.

Recall that we have the following homeomorphism:

$$\begin{aligned} \operatorname{Spec}(U^{-1}S) &\longrightarrow \{ Q \in \operatorname{Spec} S \mid Q \cap U = \emptyset \} \\ Q(U^{-1}S) &\longleftarrow\!\shortmid \; Q \end{aligned}$$

Notice: $Q \cap U = \emptyset \iff \varphi^{-1}(Q) \subseteq P$, so $f^{-1}(P)$ is contained in

For $Q \in f^{-1}(P)$, we need $P \subseteq \varphi^{-1}(Q)$ as well.

$P \subseteq \varphi^{-1}(Q) \iff \varphi(P) \subseteq Q \iff PS \subseteq Q \iff P(U^{-1}S) \subseteq Q(U^{-1}S)$

So $f^{-1}(P) \cong V(P(U^{-1}S)) \subseteq \operatorname{Spec}(U^{-1}S)$.
Which is homeomorphic to $\operatorname{Spec}\left(U^{-1}S / PU^{-1}S\right)$.

Thus, we just need $U^{-1}S / PU^{-1}S \cong R_P / PR_P \otimes S$

We know that $R_P \otimes_R S \to U^{-1}S$ is an isomoprphism.

Tensoring the SES

$$0 \to PR_P \longrightarrow R_P \longrightarrow R_P/PR_P \longrightarrow 0$$

yields

$$\begin{array}{ccccccc} PR_P \otimes S & \xrightarrow{\alpha} & R_P \otimes S & \longrightarrow & R_P/PR_P \otimes S & \longrightarrow & 0 \\ & & \cong & & & & \\ & & U^{-1}S & & & & \end{array}$$

The image of α is generated by elts of the form $\frac{p}{v} \otimes s$, for $p \in P$, $v \notin P$, $s \in S$.

These correspond to elts of $U^{-1}S$ of the form $\frac{\varphi(p)s}{u}$, where $u \in \varphi(R \setminus P) = U$, which generate $PU^{-1}S$.

Thus, we get desired isomorphism. $\square$

So we now have a nice way to describe fibers.

Ex: If $\mathbb{C}[x] \hookrightarrow \mathbb{C}[x,y]$ is the inclusion, and f the induced map on Spec, what does the fiber over $P \in \operatorname{Spec} \mathbb{C}[x]$ look like?

If $P = (x-a)$, $k(P) = \left(\mathbb{C}[x]_P / P_P\right) \cong \mathbb{C}[x]/(x-a)$

so $\mathbb{C}[x,y] \otimes_{\mathbb{C}[x]} k(P) \cong \mathbb{C}[x,y] \otimes_{\mathbb{C}[x]} \mathbb{C}[x]/(x-a)$

$$\cong \mathbb{C}[x,y]/(x-a) \cong \mathbb{C}[y]$$

If $P=(0)$, $k(0) = \mathbb{C}[x]_{(0)}/(0)$ = field of fractions of $\mathbb{C}[x]$

so $\mathbb{C}[x,y] \otimes_{\mathbb{C}[x]} k(0)$

$$\cong \underbrace{\mathbb{C}(x)}_{\text{field of fractions of } \mathbb{C}[x]}[y]$$

Ex: Consider $\varphi: \mathbb{C}[x] \longrightarrow \mathbb{C}[x,y]/(y^2-x)$, the inclusion,

and f the induced map on Spec.

Then if $P=(x-a)$, we have

$$\mathbb{C}[x,y]/(y^2-x) \otimes_{\mathbb{C}[x]} \mathbb{C}[x]/(x-a)$$

$$\cong \mathbb{C}[x,y]/(y^2-x,\, x-a) \cong \mathbb{C}[y]/(y^2-a)$$

$$\operatorname{Spec} \mathbb{C}[y]/(y^2-a) = \{(y-a),(y+a)\} \text{ if } a \neq 0$$

$$\text{or } \{(y)\} \text{ if } a=0.$$

If $P=(0)$,

$$\mathbb{C}[x,y]/(y^2-x) \otimes_{\mathbb{C}[x]} \mathbb{C}[x]_{(0)}$$

The only prime ideal in $\mathbb{C}[x,y]$ that contains (y^2-x) but doesn't meet $\mathbb{C}[x]$ is (y^2-x) itself.

So Spec of this ring is a single point.

Ex: $f: \operatorname{Spec}\mathbb{Q} \longrightarrow \operatorname{Spec}\mathbb{Z}$ has fiber over (0):

$$\operatorname{Spec}\Big(\underbrace{\mathbb{Q}\otimes_{\mathbb{Z}}\mathbb{Z}_{(0)}}_{\mathbb{Q}}\Big) = \text{one point}$$

over (p), $\operatorname{Spec}\Big(\underbrace{\mathbb{Q}\otimes\mathbb{Z}/(p)}_{0}\Big) = \emptyset.$